



The Approximation Properties of the Numerical Scheme of the Black-Schole Equation with Volatile Portfolio Risk Measure

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Abstract

We study the numerical approximation in space of the solution of Black-Schole's equation with volatile portfolio risk measure. Making use of the L^2 theorem of solvability in Sobolev spaces, the solution is approximated in space, with finite –difference methods.

Keywords: Sobolev space, Non-linear black-scholes equation, Transaction cost, Portfolio risk measure, Finite difference methods.

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1. Introduction

The numerical methods and possible approximation results are strongly linked to the theory on the solvability of the PDEs. In the present paper, we make use of the L^2 theory of solvability of linear PDEs in Sobolev spaces. The finite difference method for approximating PDE is a well developed area, which has been extensively researched since the first half of the last century. We refer to Thomee [1] for a brief summary of the method's history, and also for the references of the seminal work by R. Courant, K. O. Friedrichs and H. Lewy, and of further major contributions by other authors. In particular, a general approach of the numerical approximation, making use of finite difference, of the Cauchy problem for a multidimensional linear parabolic PDE of order $M \geq 2$, with bounded time and space-dependent coefficients, can be found in Thomee [1]. This approach is pursued under a strong setting, where the PDE problem has a classical solution. Also Zhao and Zhang [2] discussed the primal-dual large-update interior-point algorithm for semi-definite optimization based on a new kernel function.

The finite difference method was also early applied to financial option pricing, the pioneering work being due to M. Brennan and E. S. Schwartz in 1978, and was, since then, widely researched in the context of the financial application, and extensively used by practitioners. For the references of the original publications and further major research, we refer to the review paper by Broadie and Detemple [3]. Most studies concerning the numerical approximation of PDE problems in Finance consider the particular case where the PDE coefficients are constant, see, e.g., [4-7]. This occurs, namely, in option pricing under the Black-Scholes model (in one or several dimensions), when the the asset price vector's drift and volatility are taken constant. The simpler PDE, with constant coefficients, is obtained after a standard change of variables, see, e.g., [8] for the one-dimensional case, and Goncalves and Grossinho [9] for the multidimensional case. The PDEs arising from the generalized option pricing model pose three challenges to the numerical approximation: the degeneracy of the equation, the coefficients being time and space-dependent and also unbounded in the space variables. In the present article, we assume that the equation is non degenerate and the coefficients are bounded, and deal only with the coefficient time and space-dependency. This paper studied the equation below on the strip $[0, T] \times \mathbb{R}^d$, with T a positive constant, we study the approximation properties of the numerical scheme of the PDE

$$L_h u - u_t + f_h = 0 \text{ in } Q(h), \quad u(0, x) = g_h(x) \text{ in } Z_h^d$$

where $Q(h) = [0, T] \times Z_h^d$, with $T \in (0, \infty)$, and f_h and g_h are functions such that $f_h : Q(h) \rightarrow \mathbb{R}$ and $g_h : Z_h^d \rightarrow \mathbb{R}$.

The work is based heavily on the PhD thesis and a working paper by Goncalves and Grossinho [9]. We deal with the PDE nonlinearity by treating the nonlinear term as if it was a free data, imposing regularity to it, and then using linear theory to derive the results.

2. The Model

Transaction costs as well as the volatile portfolio risk depend on the time –lag between two consecutive transactions. Minimizing their sum yields the optimal length of the hedge interval –time lag . This leads to a fully nonlinear parabolic PDE. If transaction costs are taken into account perfect replication of the contingent claim is no longer possible. Modeling the short rate $r = r(t)$ by a solution to a one factor stochastic differential equation,

$$dS = \mu(s, t)dt + \sigma(s, t)dw, \tag{2.1}$$

where $\mu(S, t)dt$ represent a trend or drift of the process and $\sigma(S, t)$ represents volatility part of the process, the risk adjusted Black-Scholes equation can be viewed as an equation with a variable volatility coefficient

$$\partial_t V + \frac{\sigma^2(s, t)}{2} S^2 \left(1 - \mu(S \partial_s V) \right)^{\frac{1}{3}} \partial_s^2 V + rS \partial_s V - rV = 0, \tag{2.2}$$

where $\sigma^2(s, t)$ depends on a solution $V = V(s, t)$ and $\mu = 3 \left(\frac{C^2 R}{2\pi} \right)^{\frac{1}{3}}$, since

$$\sigma^2(s, t) = \sigma^2 \left(1 - \mu(S \partial_s^2 V(S, t)) \right)^{\frac{1}{3}}.$$

Incorporating both transaction costs and risk arising from a volatile portfolio into Equation (2.2) we have the change in the value of portfolio to become.

$$\partial_t V + \frac{\sigma^2(s, t)}{2} S^2 \partial_s^2 V + rS \partial_s V - rV = (r_{TC} + r_{VP})S, \tag{2.3}$$

where $r_{TC} = \frac{C|\Gamma|\sigma S}{\sqrt{2\pi}} \frac{1}{\sqrt{\Delta t}}$ is the transaction costs measure, $r_{VP} = \frac{1}{2} R \sigma^4 S^2 \Gamma^2 \Delta t$ is the volatile portfolio risk measure and $\Gamma = \partial_s^2 V$. Minimizing the total risk with respect to the time lag Δt yields as in Osu and Olunkwa [10];

$$\min_{\Delta t} (r_{TC} + r_{VP}) = \frac{3}{2} \left(\frac{C^2 R}{2\pi} \right)^{\frac{1}{3}} \sigma^2 |S \partial_s^2 V|^{\frac{4}{3}}.$$

For simplicity of solution and without loss of generality, we choose the minimized risk as

$$\left\{ \min_{\Delta t} (r_{TC} + r_{VP}) \right\}^{\frac{3}{2}} = A s^2 \partial_s^2 v, \tag{2.4a}$$

with

$$A = \left(\frac{3}{2}\right)^{\frac{3}{2}} \left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{2}} \sigma^3. \quad (2.4b)$$

They change in the value of the portfolio after minimizing the total risk with respect to time lag is given as

$$\partial_t V + \frac{\sigma^2 (1 - \mu(S \partial_s^2 v(s, t)))^{\frac{1}{3}}}{2} S^2 \partial_s^2 V + rS \partial_s V - rV = As^2 \partial_s^2 v, \quad (2.5)$$

which can also be written as

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} (1 - \mu(S^2 \partial_s^2 V(S, t)))^{\frac{1}{3}} s^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = As^2 \frac{\partial^2 V}{\partial S^2}.$$

The left hand of Equation 2.5 is the usually Black-Sholes formula. Setting

$$S = e^x, V(x, y) = u(e^x, t) \text{ and } h(e^x) = g(x)$$

we have Equation (4) becoming;

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} (1 - \mu(S^2 \partial_s^2 U(S, t)))^{\frac{1}{3}} s^2 \frac{\partial^2 u}{\partial x^2} + r \frac{\partial u}{\partial x} - ru = A \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \right),$$

which implies;

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} (1 - \mu(S^2 \partial_s^2 u(e^x, x)))^{\frac{1}{3}} \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \right) + r \frac{\partial u}{\partial x} - ru = A \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \right). \quad (2.6)$$

Let $k = \frac{\sigma^2}{2} (1 - \mu(S^2 \partial_s^2 u(e^x, x)))^{\frac{1}{3}}$, then Equation(5) reduces to

$$\frac{\partial u}{\partial t} + k \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \right) + r \frac{\partial u}{\partial x} - ru = A \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \right), \quad (2.7)$$

or

$$\frac{\partial U}{\partial t} + k \frac{\partial^2 U}{\partial x^2} + (r - k) \frac{\partial U}{\partial x} - rU = A \left| \frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial x} \right|, \quad (2.8)$$

which is equivalent to that in Maraiam, et al. [11]. We further assume that there is no accumulated interest on the portfolio. Hence $r = 0$ and the new portfolio becomes

$$-\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} + k \frac{\partial u}{\partial x} = -A \left| \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \right|$$

or

$$-\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} + k \frac{\partial u}{\partial x} = A \left| \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \right| \quad (2.9)$$

with the initial condition

$$U(x, 0) = \max(1 - e^{-x}, 0)$$

$$-A = \bar{A}.$$

$$-\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} + k \frac{\partial u}{\partial x} = \bar{A} F \left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \right) \quad \text{in } Q \quad (2.10)$$

$$u(x, 0) = g(x) \quad \text{in } R^d$$

Equation (2.10) can be written as

$$-k \frac{\partial^2 u}{\partial x^2} + k \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} + AF \left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \right) = 0 \quad \text{in } Q \quad (2.11)$$

$$u(x, 0) = g \quad \text{in } R^d$$

Let $Lu = \bar{k} \frac{\partial^2 u}{\partial x^2} + k \frac{\partial u}{\partial x}$, $u_t = -\frac{\partial u}{\partial t}$ and $f = +AF \left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \right)$

where $\bar{k} = -k$

Equation 2.11 can be written as

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0, x) = g(x) \text{ in } R^d \quad (2.12)$$

with

$$L(t, x) = \bar{k}(t, x) \frac{\partial^2}{\partial x \partial x} + k(t, x) \frac{\partial}{\partial x}. \tag{2.13}$$

3. Numerical Approximations in Space: The Discrete Framework

In this section, we discretize in space problem (2.12), with the use of a finite-difference scheme. By considering suitable discrete function space, we can show that the discrete framework we set is a particular case of the general framework, therefore holding an existence and uniqueness result for the solution of the discretized problem. We define the h -grid on \mathbb{R}^d , with $h \in (0, 1]$

$$Z_h^d = \left\{ x \in \mathbb{R}^d : x = h \sum_{i=1}^d e_i n_i, \quad n_i = 0, \pm 1, \pm 2, \dots \right\}.$$

Denote

$$\partial^+ u = \partial^+ u(t, x) = h^{-1} (u(t, x + he_i) - u(t, x))$$

and

$$\partial^- u = \partial^- u(t, x) = h^{-1} (u(t, x) - u(t, x - he_i)),$$

the forward and backward difference quotients in space, respectively. Define the discrete operator

$$L_h(t, x) = \bar{k}(t, x) \partial^- \partial^+ + k(t, x) \partial^+$$

We consider the discrete problem

$$L_h u - u_t + f_h = 0 \text{ in } Q(h), u(0, x) = g_h(x) \text{ in } Z_h^d, \tag{3.1}$$

where $Q(h) = [0, T] \times Z_h^d$, with $T \in (0, \infty)$, and f_h and g_h are functions such that $f_h : Q(h) \rightarrow \mathbb{R}$ and $g_h : Z_h^d \rightarrow \mathbb{R}$.

Consider function $v : Z_h^d \rightarrow \mathbb{R}$. We introduce the 0-order discrete Sobolev space

$$l^{0,2} = \{v : Z_h^d \rightarrow \mathbb{R} : |v|_{l^{0,2}} < \infty\},$$

where the norm $|v|_{l^{0,2}}$ is defined by

$$|v|_{l^{0,2}} = \left(\sum_{x \in Z_h^d} |v(x)|^2 h^d \right)^{1/2}$$

Define the inner product

$$(v, u)_{l^{0,2}} = \sum_{x \in Z_h^d} v(x) u(x) h^d,$$

for any $v, w \in l^{0,2}$, which induces the above norm.

It could be checked trivially that $(\cdot, \cdot)_{l^{0,2}}$ and $|\cdot|_{l^{0,2}}$, as defined above, are an inner product and norm, respectively. We show next the good structure of $l^{0,2}$.

Proposition 3.1. The function space $l^{0,2}$, is a Hilbert space.

Proof: To prove that $l^{0,2}$ is a Hilbert space we have that the inner product space $l^{0,2}$ is complete, i.e., that any Cauchy sequence in $l^{0,2}$ is convergent in the space norm.

Let (v_n) be a Cauchy sequence in $l^{0,2}$, i.e., for all $\varepsilon > 0$ exists N such that for $m, n > N$

$$|v_m - v_n|_{l^{0,2}} = \left(\sum_{x \in Z_h^d} |v_m(x) - v_n(x)|^2 h^d \right)^{1/2} < \varepsilon. \tag{3.2}$$

Therefore, for every $x \in Z_h^d$

$$|v_m(x) - v_n(x)|^2 h^d < \varepsilon^2, \text{ for } m, n > N. \tag{3.3}$$

Let us fix $x = x_0$. From (3.3), we see that $(v_1(x_0), v_2(x_0), \dots)$ is a Cauchy sequence of real numbers, therefore convergent. Write $v_m(x_0) \rightarrow v(x_0)$. Using these limits, we define $v = v(x)$, for each $x \in Z_h^d$.

Let B be a ball in Z_h^d . From (3.3), for $m, n > N$

$$\sum_{x \in B} |v_m(x) - v_n(x)|^2 h^d < \varepsilon^2.$$

Letting $n \rightarrow \infty$, for $m > N$

$$\sum_{x \in B} |v_m(x) - v(x)|^2 h^d \leq \varepsilon^2$$

Letting now the diameter of B go to ∞ , for $m > N$

$$\sum_{x \in Z_h^d} |v_m(x) - v(x)|^2 h^d \leq \varepsilon^2 \tag{3.4}$$

Inequalities (3.4) implies that $v_m - v \in l^{0,2}$. Finally, (3.4) also implies that $v_m \rightarrow v$, and the result is proved. For functions $v : Z_h^d \rightarrow \mathbf{R}$. We introduce also the discrete Sobolev space of order 1

$$l^{1,2} = \{v : Z_h^d \rightarrow \mathbf{R} : |v|_{l^{1,2}} < \infty\}$$

With the norm $|v|_{l^{1,2}}$ defined by

$$|v|_{l^{1,2}} = \left(|v|_{l^{0,2}}^2 + |\partial^+ v|_{l^{0,2}}^2 \right)^{1/2}$$

Let us endow this function space with the inner product, generating the above norm,

$$(v, w)_{l^{1,2}} = (v, w)_{l^{0,2}} + (\partial^+ v, \partial^+ w)_{l^{0,2}}$$

Where v, w are any functions in $l^{1,2}$.

To show that that the discrete framework we set a particular case of the general framework, we begin by checking that $l^{1,2}$ is a reflexive space and separable Banach space, continuously and densely embedded into the Hilbert space $l^{0,2}$. Following the same steps as in proof of proposition 1, it could be easily proved that $l^{1,2}$ is a complete inner product space. Therefore $l^{1,2}$ is reflexive. We state that $l^{1,2}$ is separable.

Proposition 3.2: The function space $l^{1,2}$ is separable, See, e.g., [9] for prove.

We can now check that $l^{1,2}$ is continuously and densely embedded in $l^{0,2}$. The continuity follows immediately from

$$|v|_{l^{0,2}} \leq |v|_{l^{1,2}} \quad \text{for all } v \in l^{1,2}$$

For the denseness, we prove the following result:

Proposition 3.3: The function space $l^{1,2}$ is densely embedded in $l^{0,2}$.

Proof: We want to prove that $\overline{l^{1,2}} = l^{0,2}$. Let us take an arbitrary function $v \in l^{0,2}$. Let B be a ball in Z_h^d . we consider the function w such that

$$w(x) = \begin{cases} v(x), & x \in B \\ 0, & \text{otherwise.} \end{cases}$$

This function belongs obviously to $l^{1,2}$. Furthermore, for any given $\varepsilon > 0$

$$|v - w|_{l^{0,2}} < \varepsilon,$$

If the diameter of B is chosen sufficiently large, the result is proved.

Now, we change point of view and consider the function $w : \mathbf{Q}(h) \rightarrow \mathbf{R}$ as functions in $[0, T]$ with values in \mathbf{R}^∞ , defined by $w(t) = \{w(t, x) : x \in Z_h^d\}$, for all $t \in [0, T]$. For these functions, we consider the subspaces $C([0, T]; l^{0,2})$ and

$$L^2([0, T]; l^{1,2}) = \{w : [0, T] \rightarrow l^{1,2} : |w|_{L^2} < \infty\}$$

With $|w|_{L^2}^2 = \int_0^T |w|_{l^{1,2}}^2 dt$.

Assumption 1. Let $M \geq 0$ be an integer.

1. There exists a constant $\lambda > 0$ such that

$$\sum_{i,j} \bar{k}(t, x) \xi^i \xi^j \geq \lambda \sum_{i=1}^d |\xi^i|^2,$$

For all $t \geq 0, x \in \mathbf{R}^d, \xi \in \mathbf{R}^d$;

2. The coefficients in L and their derivatives in x up to the order m are measurable functions in $[0, T] \times \mathbf{R}^d$ such that

$$|D_x^\alpha \bar{k}| \leq K \quad \forall |\alpha| \leq m, \quad |D_x^\alpha k| \leq K$$

For any $t \in [0, T], x \in \mathbf{R}^d$, with K a constant and D_x^α denoting the α^{th} partial derivative operator with respect to x ;

$$3. f \in L^2([0, T]; W^{m-1,2}), g \in W^{m,2}.$$

Notation. For $m = 0$, we use the notation $W^{m-1,2} = W^{-1,2} = (W^{1,2})^*$, where $(W^{1,2})^*$ is the dual of $W^{1,2}$.

We make some assumption over the regularity of the data f_h and g_h in (3.1)

Assumption 2. We assume

$$1. f_h \in L^2([0, T]; l^{0,2}),$$

$$2. g_h \in l^{0,2}$$

Remark 3.1: In the above assumption 2,(1) can be replaced by the weaker assumption

$$f_h \in L^2([0, T]; (l^{1,2})^*), \text{ where } (l^{1,2})^* \text{ denotes dual space of } l^{1,2} .$$

Remark 3 .The boundedness of the difference quotient of the difference quotient

$$\partial^+ \bar{k} = \partial^+ \bar{k}(t, x) = h^{-1}(\bar{k}(t, x + he_i) - \bar{k}(t, x))$$

can be obtained from (2) in Assumption 1. In fact ,

$$|\partial^+ \bar{k}(t, x)| = |h^{-1}(\bar{k}(t, x + he_i) - \bar{k}(t, x))| \leq \left| \frac{\partial}{\partial x} \bar{k}(t, x + re_i) \right|,$$

for some τ such that $0 < \tau < h$. Thus $\left| \frac{\partial}{\partial x} \bar{k} \right| \leq K$ implies $|\partial^+ \bar{k}| \leq K$

We define the generalized solution of problem (10).

Definition 3. We say that $u \in C([0, T]; l^{0,2}) \cap L^2([0, T]; l^{1,2})$ is a generalized solution (3.1) if for all $t \in [0, T]$

$$(u(t), \varphi) = (g_h, \varphi) + \int_0^t \{ -(\bar{k}(s) \partial^+ u(s), \partial^+ \varphi) + (k(s) \partial^+ u(s) - \partial^+ \bar{k}(s) \partial^+ u(s), \varphi) + \langle f_h(s), \varphi \rangle \} ds$$

holds for all $\varphi \in l^{1,2}$.

Notation. In the above definition, and in the sequel, (\cdot, \cdot) denotes the inner product in $l^{0,2}$.

Theorem 3.1: Under (1)-(3) in assumption 1, problem (2.12) admits a unique generalised solution u on $[0, T]$ More over $u \in C([0, T]; W^{m,2}) \cap L^2([0, T]; W^{m+1,2})$ and

$$\sup_{0 \leq t \leq T} |u(t)|_{W^{m,2}}^2 + \int_0^T |u(t)|_{W^{m+1,2}}^2 dt \leq N \left(|g|_{W^{m,2}}^2 + \int_0^T |f(t)|_{W^{m-1,2}}^2 dt \right)$$

With N a constant.

We prove next an existence and uniqueness result for the solution of the discrete problem (2.10) ,providing ,in addition ,an estimate for the solution .With this result ,we show that the numerical scheme is stable ,ie ,informally ,that the discrete problem's solution remains bounded independently of space-step h .The result is obtained as consequence of Theorem 1,remaining only to show that ,within the discrete framework we constructed, (1)-(3) in assumption 1 holds.

Theorem 3.2: Under (1)-(3) in Assumption 2 ,problem (3.1) admits a unique generalized solution on $[0, T]$.Moreover

$$\sup_{0 \leq t \leq T} |u(t)|_{l^{0,2}}^2 + \int_0^T |u(t)|_{l^{1,2}}^2 dt \leq N \left(|g_h|_{l^{0,2}}^2 + \int_0^T |f_h(t)|_{l^{0,2}}^2 dt \right),$$

With N a constant independent of h .

Proof. Let $L_h(s): l^{1,2} \rightarrow (l^{1,2})^*$ and define for all $\varphi, \psi \in l^{1,2}$

$$\langle L_h(s) \psi, \varphi \rangle := -(\bar{k}(s) \partial^+ \psi, \partial^+ \varphi) + (k(s) \partial^+ \psi - \partial^+ \bar{k}(s) \partial^+ \psi, \varphi)$$

It suffices to prove the estimates:

1. $\exists K, \lambda > 0$ constant : $\langle L_h(s) \psi, \psi \rangle \leq K |\psi|_{l^{0,2}}^2 - \lambda |\psi|_{l^{1,2}}^2 \quad \forall \psi \in l^{1,2}; ;$
2. $\exists K$ constant : $|\langle L_h(s) \psi, \varphi \rangle| \leq K |\psi|_{l^{1,2}} \cdot |\varphi|_{l^{1,2}} \quad \forall \psi, \varphi \in l^{1,2}$.

For the first property, omitting the variable $x \in Z_h^d$ in the writing, we have

$$\begin{aligned} \langle L_h(s) \psi, \psi \rangle &= - \sum_{x \in Z_h^d} \bar{k}(s) \partial^+ \psi \partial^+ \psi h^d + \sum_{x \in Z_h^d} (k(s) - \partial^+ \bar{k}(s)) \partial^+ \psi \psi h^d \\ &+ \sum_{x \in Z_h^d} c(s) \psi \psi h^d \leq -\lambda \sum_{x \in Z_h^d} |\partial^+ \psi|^2 h^d + 2K \sum_{x \in Z_h^d} |\partial^+ \psi \psi| h^d + K \sum_{x \in Z_h^d} |\psi|^2 h^d \\ &= -\lambda |\partial^+ \psi|_{l^{0,2}}^2 + 2k \sum_{x \in Z_h^d} |\partial^+ \psi \psi| h^d + K |\psi|_{l^{0,2}}^2. \end{aligned} \tag{3.7}$$

Owning to (1) and (2) in Assumption 1. Applying the Cauchy's inequality with ε to the second term of last member in (3.7), we obtain

$$\begin{aligned} |\langle L_h(s) \psi, \psi \rangle| &= -\lambda |\partial^+ \psi|_{l^{0,2}}^2 + \varepsilon K \sum_{x \in Z_h^d} |\partial^+ \psi|^2 h^d + \frac{K}{\varepsilon} \sum_{x \in Z_h^d} |\psi|^2 h^d + K |\psi|_{l^{0,2}}^2 \\ &= -\lambda |\partial^+ \psi|_{l^{0,2}}^2 - |\psi|_{l^{0,2}}^2 + \varepsilon K |\partial^+ \psi|_{l^{0,2}}^2 + \frac{K}{\varepsilon} |\psi|_{l^{0,2}}^2 \\ &+ (K + \lambda) |\psi|_{l^{0,2}}^2 \leq -\lambda |\psi|_{l^{0,2}}^2 + K |\psi|_{l^{0,2}}^2 \end{aligned}$$

with $\lambda > 0$, K constants, by taking ε sufficiently small and the first property is proved

$$\begin{aligned} |\langle L_h(s)\psi, \phi \rangle| &= \left| - \sum_{x \in Z_h^d} \bar{k}(s) \partial^+ \psi \partial^+ \phi h^d + \sum_{x \in Z_h^d} k(s) \partial^+ \psi \phi h^d - \sum_{x \in Z_h^d} c(s) \psi \phi h^d \right| \leq K \sum_{x \in Z_h^d} |\partial^+ \psi \partial^+ \phi| h^d \\ &+ K \sum_{x \in Z_h^d} |\partial^+ \psi \phi| h^d + K \sum_{x \in Z_h^d} |\psi \phi| h^d \leq K |\partial^+ \psi|_{l,0,2} |\partial^+ \phi|_{l,0,2} \\ &+ K |\partial^+ \psi|_{l,0,2} |\phi|_{l,0,2} + K |\psi|_{l,0,2} |\phi|_{l,0,2} \leq K |\psi|_{l,1,2} |\phi|_{l,1,2} \end{aligned}$$

where the above writing convention is kept. Owing to theorem 1 the result follows.

4. Approximation Proprieties

In this section, we study the approximation proprieties of the numerical scheme (3.1).

Theorem 4.1: Let m be an integer strictly greater than

$d/2$. Let $u(t) \in W^{m+2,2}, v(t) \in W^{m+3,2}$, for all $t \in [0, T]$ Then there exists a constant N independent of h such that

$$\begin{aligned} (1) \sum_{x \in Z_h^d} |u_{x^i}(t, x) - \partial_i^+ u(t, x)|^2 h^d &\leq h^2 N |u(t)|_{W^{m+2,2}}^2, \\ (2) \sum_{x \in Z_h^d} |u_{x^i x^j}(t, x) - \partial_j^- \partial_i^+ v(t, x)|^2 h^d &\leq h^2 N |v(t)|_{W^{m+3,2}}^2, \end{aligned}$$

.for all $t \in [0, T]$

In order to prove theorem 4.1, we state two results. We recall a fundamental theorem on the embedding of $W^{m,2}(U)$ into better spaces, see, eg., [9].

Theorem 4.2: (Sobolev embedding theorem). Let U be a bounded domain in

Let $v \in W^{m,2}(U)$. If $m > \frac{d}{2}$ then $v \in C^{(m - [\frac{d}{2}] - 1) + \delta}(U)$, where

$$\delta = \begin{cases} [\frac{d}{2}] + 1 - \frac{d}{2}, & \text{if } \frac{d}{2} \text{ is not an integer} \\ \text{any positive number } < 1, & \text{if } \frac{d}{2} \text{ is an integer} \end{cases}$$

Moreover $|v|_{(m - [\frac{d}{2}] - 1) + \delta; U} \leq N |v|_{W^{m,2}(U)}$, with N a constant depending only on m, d, δ and U . Notation. We use

the notation $|v|_{k+\delta; U}$ for the norm of v in the Holder space $C^{k+\delta}(U)$ ($0 < \delta < 1, k = 0, 1, \dots$).

We also recall the following propriety of Sobolev space (see, eg [12])

Proposition 4.1: Let $v \in W^{m,2}(U)$. If V is an open subset of U , then $v \in W^{m,2}(V)$. We now prove Theorem 4.1.

Proof .: (Theorem 4.1) Let us prove (??). By the mean-value theorem,

$$\partial_i^+ u(t, x) = h^{-1} (u(t, x + he_i) - u(t, x)) = u_{x^i}(t, x + \theta he_i)$$

and

$$u_{x^i}(t, x) - \partial_i^+ u(t, x) = u_{x^i}(t, x) - u_{x^i}(t, x + \theta he_i) = \theta h u_{x^i x^i}(t, x + \theta' he_i),$$

For some $0 < \theta' < \theta < 1$, we consider d -cells

$$R_h = \{(x^1, x^2, \dots, x^d) \in \mathbb{R}^d : x_h^i < x^i x_h^i + h, i = 1, 2, \dots, d\}$$

With $x_h = (x_h^1, x_h^2, \dots, x_h^d) \in Z_h^d$ fixed.

$$|u_{x^i x^i}(t, x_h + \theta' he_i)| \leq \sup_{x \in R_h} |u_{x^i x^i}(t, x)|,$$

and then

$$|u_{x^i}(t, x_h) - \partial_i^+ u(t, x_h)|^2 \leq h^2 \sup_{x \in R_h} |u_{x^i x^i}(t, x)|^2. \tag{4.1}$$

Let us consider the particular d -cell where $h = 1$ and $x_h = x_1 = (0, 0, \dots, 0)$, denotes it by R_1^0 . We have

$$\sup_{x \in R_h} |u_{x^i x^i}(t, x)| = \sup_{x \in R_1^0} |u_{x^i x^i}(t, x_h + hx)|. \tag{4.2}$$

Take open balls $B_h \supset R_h$, such that the vertices $x_h^i, x_h^i + h, i = 1, 2, \dots, d$, of the d -cell lie on the limiting sphere. Denote B_1^0 the ball containing R_1^0 we have

$$\sup_{x \in B_1^0} |u_{x_i x_i}(t, x_h + hx)|^2 \leq \sup_{x \in B_1^0} |u_{x_i x_i}(t, x_h + hx)|^2 \tag{4.3}$$

Taking in mind proposition 4.1, as $B_1^0 \subset U$ is an bounded domain of class C^1 , the hypotheses of theorem 4 are satisfied and for $m \geq d/2$,

$$\begin{aligned} \sup_{x \in B_1^0} |u_{x_i x_i}(t, x_h + hx)|^2 &\leq N \sum_{|\alpha| \leq m} \int_{B_1^0} |D_x^\alpha u_{x_i x_i}(t, x_h + hx)|^2 dx \\ &\leq N \sum_{|\alpha| \leq m+2} \int_{B_1^0} |D_x^\alpha u(t, x_h + hx)|^2 dx \\ &= N \sum_{|\alpha| \leq m+2} \int_{B_1^0} |D_x^\alpha u(t, x)|^2 h^{-d} h^{2|\alpha|} dx \\ &\leq N \sum_{|\alpha| \leq m+2} \int_{B_1^0} |D_x^\alpha u(t, x)|^2 h^{-d} dx \end{aligned} \tag{4.4}$$

Then ,by (4.1),(4.2),(4.3) and (4.4),owing to the particular geometry of the framework we have set, we finally obtain

$$\begin{aligned} &\sum_{x_h \in Z_h^d} |u_{x_i}(t, x_h) - \partial_i^+ u(t, x_h)|^2 h^d \\ &\leq h^2 N \sum_{|\alpha| \leq m+2} \sum_{x_h \in Z_h^d} \int_{B_h(x_h)} |D_x^\alpha u(t, x)|^2 dx \leq 2h^2 N \sum_{|\alpha| \leq m+2} \sum_{x_h \in Z_h^d} \int_{R_h(x_h)} |D_x^\alpha u(t, x)|^2 dx \\ &\leq h^2 N |u(t)|_{W^{m+2,2}}^2, \end{aligned}$$

where $B_h(x_h) \ni B_h, R_h(x_h) \ni R_h$, and the proof for (??) is complete. The proof for the theorem 4.2 is similar.

Finally, owing to the stability and consistency proprieties of the numerical scheme, we prove the convergence of the discrete problem's solution to the exact problem's solution and compute a convergence rate. The accuracy obtained is order 1.

Theorem 4.3: let the hypotheses of Theorems 4.1 and 4.2 be satisfied .Let m be an integer strictly greater than $d/2$, and denote by u the solution of (2.12) in Theorem 4.1 and by u_h the solution of (3.1) in Theorem 2. Assume also that $u \in L^2([0, T]; W^{m+3,2})$ Then

$$\sup_{0 \leq t \leq T} |u(t) - u_h(t)|_{l^{0,2}}^2 + \int_0^T |u(t) - u_h(t)|_{l^{0,2}}^2 dt \leq h^2 N \int_0^T |u(t)|_{W^{m+3,2}}^2 dt + N \left(|g - g_h|_{l^{0,2}}^2 + \int_0^T |f(t) - u_h(t)|_{l^{0,2}}^2 dt \right),$$

for some constant N independent of h .

$$\begin{cases} L_h(u - u_h) - \frac{d}{dt}(u - u_h) + (L - L_h)u + (f - f_h) = 0 & \text{in } Q(h) \\ (u - u_h)(0, x) = (g - g_h)(x) & \text{in } Z_h^d. \end{cases} \tag{4.5}$$

We have that $(f - f_h) \in L^2([0, T]; l^{0,2})$ and $(g - g_h) \in l^{0,2}$, obviously .With respect to the term $(L - L_h)u$, note that

$$\sum_{x \in Z_h^d} |(L - L_h)u(t)|^2 h^d = \sum_{x \in Z_h^d} \left| k(t, x) \left(\frac{\partial^2}{\partial x^2} - \partial^- \partial^+ \right) u(t, x) + \bar{k}(x, t) \left(\frac{\partial}{\partial x} - \partial^+ \right) u(t, x) \right|^2 h^d < \infty$$

If $u \in W^{m+3,2}$, for all $t \in [0, T]$, owing to (2) in Assumption 2 and to theorem 4.3. In consequence, we also have that $(L - L_h)u \in L^2([0, T]; l^{0,2})$

We have shown that problem (4.5) satisfies the hypotheses of Theorem 2, therefore the following estimate holds

$$\sup_{0 \leq t \leq T} |u(t) - u_h(t)|_{l^{0,2}}^2 + \int_0^T |u(t) - u_h(t)|_{l^{1,2}}^2 dt \leq N \left(|g - g_h|_{l^{0,2}}^2 + \int_0^T |f(t) - f_h(t)|_{l^{0,2}}^2 dt + \int_0^T |(L - L_h)u(t)|_{l^{1,2}}^2 dt \right).$$

Owing again to (2) in Assumption 1 and to Theorem 4.3, the result follows.

Corollary 4.1: Let the hypotheses of Theorem 4.3 be satisfied and denote by u the solution of (2.12) in Theorem 1 and u_h the solution of (3.1) in Theorem 4.2. If there is a constant N independent h independent of h such that

$$|g - g_h|_{l^{0,2}}^2 + \int_0^T |f(t) - f_h(t)|_{l^{0,2}}^2 dt \leq h^2 N \left(|g|_{W^{m,2}}^2 + \int_0^T |f(t)|_{W^{m-1,2}}^2 dt \right),$$

then

$$\sup_{0 \leq t \leq T} |u(t) - u_h(t)|_{l^{0,2}}^2 + \int_0^T |u(t) - u_h(t)|_{l^{1,2}}^2 dt \leq h^2 N \left(\int_0^T |u(t)|_{W^{m+3,2}}^2 dt + |g|_{W^{m,2}}^2 + \int_0^T |f(t)|_{W^{m-1,2}}^2 dt \right).$$

Proof: The result follows immediately from Theorem 4.3.

5. Conclusion

Wang and Zeng [13] first use central difference scheme to discretize the nonlinear partial differential equation and later use Newton iteration method to solve the nonlinear system of equations. We discretize in space problem (2.12), with the use of a finite-difference scheme. By considering suitable discrete function space, we can show that the discrete framework we set is a particular case of the general framework, therefore holding an existence and uniqueness result for the solution of the discretized problem. We also investigate the consistency of the numerical scheme and prove that the difference quotients approximate partial derivatives (with accuracy of order 1). The result is obtained under stronger regularity assumptions, and using Sobolev embedding.

References

- [1] V. Thomee, *Finite difference methods for linear parabolic equations. Handbook of Numerical Analysis I (General Editors: P. G. Ciarlet, and J. L. Lions)*. North-Holland, 1990.
- [2] D. Zhao and M. Zhang, "A primal-dual large-update interior-point algorithm for semi-definite optimization based on a new kernel function," *Stat. Optim. & Inf. Comput.*, vol. 1, pp. 41-61, 2013.
- [3] M. Broadie and J. B. Detemple, "Option pricing: Valuation models and applications," *Manag. Sci.*, vol. 50, pp. 1145-1177, 2004.
- [4] G. Barles, C. Daher, and M. Romano, "Convergence of numerical schemes for parabolic equations arising in finance theory," *Math. Models Methods Appl. Sci.*, vol. 5, pp. 125-143, 1995.
- [5] P. P. Boyle and Y. Tian, "An explicit finite difference approach to the pricing of barrier options," *Appl. Math. Finance*, vol. 5, pp. 17-43, 1998.
- [6] G. Fusai, S. Sanfelici, and A. Tagliani, "Practical problems in the numerical solution of PDE's in finance. (Editors: E. Canestrelli, M. Corazza and P. Ferretti)," *Rend. Studi Econ. Quant.*, vol. 1, pp. 34-43, 2001.
- [7] M. Gilli, E. Kellezi, and G. Pauleto, "Solving finite difference schemes arising in trivariate option pricing," *J. Econ. Dynam. Control*, vol. 26, pp. 1499-1515, 2002.
- [8] D. Lamberton and B. Lapeyre, *Introduction to stochastic calculus applied to finance*. U.K: Chapman and Hall, 1996.
- [9] F. F. Goncalves and M. R. Grossinho, "Numerical approximation of multidimensional parabolic partial differential equations arising in financial mathematics Cemapre, Iseg," Technical University of Lisbon, Rua do Quelhas 6, 1200-781 Lisboa, Portugal, 2008.
- [10] B. O. Osu and C. Olunkwa, "A solution by stochastic iteration method for nonlinear black-scholes equation with transaction cost and volatile portfolio risk in Hilbert space," *International Journal of Mathematical Analysis and Application*, vol. 1, pp. 43-48, 2014.
- [11] M. C. Maraïam, E. K. Ncheuguim, and I. Sengupta, "Solution to a nonlinear black-scholes equation," *Electronic Journal of Differential Equations*, vol. 158, pp. 1-10, 2011.
- [12] L. C. Evans, *Partial differential equations. Graduate studies in mathematics. 19*. U.S.A: American Mathematical Society, 1998.
- [13] X. Wang and T. Zeng, "Solve thermal explosion model by central difference and newton iteration method," *Stat., Optim. Inf. Comput.*, vol. 3, pp. 197-205, 2015.

Bibliography

- [1] F. Goncalves, "Numerical approximation of partial differential equations arising in financial option pricing," Ph.D. Thesis, University of Edinburgh, U.K, 2007.
- [2] I. Gyaongy and N. V. Krylov, "Stochastic partial differential equations with un-bounded coefficients and applications I," *Stochastic*, vol. 32, pp. 53-91, 1990.
- [3] J. L. Lions, *Quelques methodes de resolution des problemes aux limites non Lineaires*. Paris, France: Dunod Gauthier-Villars, 1969.